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On the role of spectral dimension in determining phase transition

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Abstract. We have studied the phase transition of the Ising model on a family of fractals called *Nice trees* whose spectral dimension d_s can take values greater than 2. The phase transition is shown to be the trivial zero-temperature one through exactly solving the free energy and the spontaneous magnetization of the system, and is different from that on Cayley trees. The result is independent of d_s of the structure and hence provides an example of trivial phase transition at $d_s \geqslant 2$, which does not agree with the argument of Yu and Gong. It suggests that the role of d_s in determining the phase transition may be complex.

1. Introduction

It is well known that on translationally invariant lattices the non-trivial phase transition is completely determined by the dimensionality D of the structure [1]. For example, $D \geqslant 2$ is the sufficient and necessary condition for the non-trivial phase transition of the Ising model. As to the situation on fractals, the role of dimensionality is replaced by ramification. Gefen and his co-workers have found that the non-trivial phase transition of the Ising model occurs if and only if the ramification of the fractal is infinite [2–4].

Recently, the point at issue comes to the rule that determines the phase transition on general structures, including translationally invariant lattices and fractals. Some authors have emphasized the importance of the spectral dimension d_s [5–9]. d_s is defined from the long-time behaviour of random walks and characterizes the anomalous feature of the dynamical behaviours [10]. Since phase transitions are governed by long-range correlations, it should be closely related to low-frequency models and d_s. It was proved by Cassi that the necessary condition for a continuous symmetry breaking is $d_s > 2$ [7]. Yu and Gong argued that when the ramification R is finite, $d_s \geqslant 2$ is the sufficient and necessary condition of a discrete symmetry breaking [8]. In both cases, structures with $d_s \geqslant 2$ are needed to check the ideas. Nice trees (NT_D, D is the dimension of the trees) are just such a kind of support whose d_s can take values greater than 2, and thus can serve as good 'laboratories' [9, 11]. Clarifying the role of d_s in determining the phase transition is the first motivation for us to study the Ising model on NT_D.

The second motivation concerns the recent work of Burioni and Cassi (BC), which found that diffusion on NT_D is non-anomalous [9]. It was generally believed that the long-time behaviour of random walks on fractals satisfied an anomalous power law, which defines the spectral dimension d_s of the structures [10]. As a result, d_s is smaller than the fractal

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dimension d_f. However, it was proved by BC that on NT_D d_s equals d_f, which agrees well with the situation on Euclidean structures, and in this sense it is non-anomalous [9]. Then the natural question is whether other statistical problems on this structure that are relevant to random walks such as phase transition also show specialities.

The third motivation comes from the similarity of the structures between NT_D and Cayley trees. It is well known that the phase transition of the Ising model on Cayley trees is of a special type called a 'continuous' phase transition [12, 13]. The spontaneous magnetization in its central portion takes a non-zero value below a finite temperature while the mean value of the magnetization of the system equals zero. Similarly, one may ask whether the situation on NT_D is also true.

In this paper we have studied the phase transition of the Ising model on NT_D. The free energy of the system is exactly solved to be analytic for all finite temperatures and the spontaneous magnetization of the original point is calculated to be zero. We conclude that the phase transition is a trivial one. This result is independent of the spectral dimension d_s and hence provides an example of trivial phase transition at $d_s \geqslant 2$, which disagrees with the argument of Yu and Gong.

2. Method and result

NT_D can be constructed in the following way: take an origin point 0 and connect it with another point 1 by a bond of length 1; to the vertex point 1, attach r branches of length 2 and all the end points are labelled 2; again to each vertex point 2 attach r branches of length 4 with the end points labelled 3 and generally attach to each vertex point n, r branches of length 2^n with the end points labelled $n+1$ (see figure 1). The fractal dimension and the spectral dimension of NT_D are given as [9]

$$d_f = d_s = 1 + \frac{\ln r}{\ln 2}. \tag{1}$$

We see that d_s can take any value by choosing r appropriately.

Consider the Ising Hamiltonian

$$-\beta E = K \sum_{(i,j)} \sigma_i \sigma_j + h \sum_i \sigma_i \qquad (K > 0) \tag{2}$$

where σ_i is the spin at the site i which takes the values ± 1, K is the coupling parameter, h is the applied field and the summation is over all the nearest neighbours. Since there is no closed loop on NT_D, we construct another set of variables $\{\sigma_0, \{\theta_\alpha\}\}$ which equals $\{\sigma_i\}$. Here σ_0 is the spin at the original point and $\theta_\alpha \equiv \sigma_i \sigma_j$ is the variable associated to the

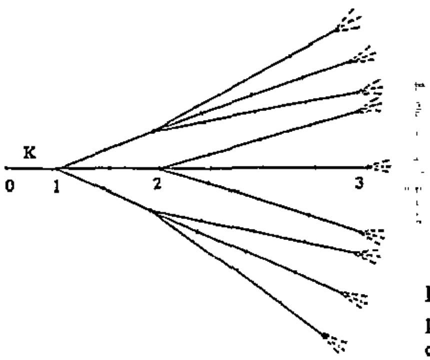

Figure 1. NT_D with the branching number $r = 3$. The branching points are labelled in the figure. The Ising model with constant coupling K is defined on the tree with the spins set at each site.

bond, where σ_i and σ_j are the spins at the ends of the bond. If no applied field exists, the partition function of the system is easily obtained as

$$Z = \sum_{\{\sigma_i\}} e^{-\beta E} = \sum_{\{\sigma_0\}} \sum_{\{\theta_\alpha\}} e^{K \sum_\alpha \theta_\alpha} = 2[2\cosh(k)]^{N_b} \tag{3}$$

where N_b is the number of bonds on NT_D. It can be easily obtained that $N_b = N_s - 1$, where N_s is the number of sites on NT_D. The free energy of the system is then given as

$$f = \lim_{N_s \to \infty} \frac{-KT \ln Z}{N_s} \simeq -KT \ln(e^K + e^{-K}) \tag{4}$$

which is analytic for all T, as is the specific heat. We should be careful in coming to the conclusion that the phase transition of the Ising model on NT_D is a trivial one. As shown in figure 2, the Ising model on NT_D is transformed through a decimation operation to a new kind of Ising model with a variety of couplings on the Cayley tree. It is well known that the Ising model with constant coupling on Cayley trees exhibits a special phase transition in the sense that the spontaneous magnetization in only a small portion of the system (e.g. the point 0 in figure 2) takes a non-zero values below a finite temperature. Thus we should also calculate the spontaneous magnetization of the original point 0 on NT_D.

Let us first introduce some useful terminology. The partition function Z of the system is separated into two parts Z^+ and Z^-, which correspond to the spin σ_0 taking the values $+1$ and -1, respectively. To each branching point s, we define the restricted partition function $\lambda_s \equiv \sum_{\{\sigma\}} e^{-\beta E}$, where the summation is over all the spins located on the subtrees started from the point s. λ_s is similarly split into λ_s^+ and λ_s^- according to the value of the spin σ_s.

With the use of the above definitions, we get the spontaneous magnetization of the point 0

$$m_0 = \lim_{h \to 0} \langle \sigma_0 \rangle = \lim_{h \to 0} \frac{1}{1 + \frac{Z^-}{Z^+}} \left(1 - \frac{Z^-}{Z^+}\right) \tag{5}$$

and

$$Z^+ = e^h (\lambda_1^+ e^K + \lambda_1^- e^{-K}) \tag{6}$$

$$Z^- = e^{-h} (\lambda_1^+ e^{-K} + \lambda_1^- e^K) \tag{7}$$

$$\frac{Z^+}{Z^-} = e^{2h} \frac{\frac{\lambda_1^+}{\lambda_1^-} e^{2K} + 1}{\frac{\lambda_1^+}{\lambda_1^-} + e^{2K}}. \tag{8}$$

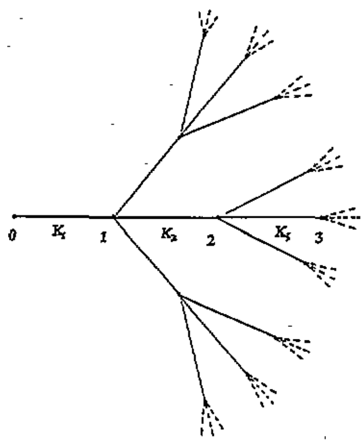

Figure 2. A branch of a Cayley tree with branching number $r = 3$. The Ising model with constant coupling on NT_D in figure 1 is transformed through the decimation operation to a new Ising model with a variety of couplings on the tree such as $K_1 = K$, $K_2 = \tanh^{-1}(\tanh^2 K)$, $K_3 = \tanh^{-1}(\tanh^4 K)$ and so on.

From the expressions (5)–(8), we note that in order to get m_0, we need to calculate λ_1^+/λ_1^- first. In the following we derive the recursion relation of λ_1^+/λ_1^-. For the path connecting the branching points s and $s+1$, we define $f_s^{++} \equiv \sum_{\{\sigma\}} e^{-\beta E}$, which corresponds to the case that the spins at the branching points s and $s+1$ both take the values +1, and the summation is over all the other spins on the path. f_s^{+-} is similarly defined for the case that the spins at the branching points take the values +1 and −1, and so is f_s^{--}. Using the functions f_s^{++}, f_s^{+-} and f_s^{--}, we have

$$\lambda_s^+ = e^h(f_s^{++}\lambda_{s+1}^+ + f_s^{+-}\lambda_{s+1}^-)^r \tag{9}$$

$$\lambda_s^- = e^{-h}(f_s^{+-}\lambda_{s+1}^+ + f_s^{--}\lambda_{s+1}^-)^r \tag{10}$$

and then

$$\frac{\lambda_s^+}{\lambda_s^-} = e^{2h}\left(\frac{\frac{\lambda_{s+1}^+}{\lambda_{s+1}^-}\frac{f_s^{++}}{f_s^{+-}} + 1}{\frac{\lambda_{s+1}^+}{\lambda_{s+1}^-} + \frac{f_s^{--}}{f_s^{+-}}}\right)^r . \tag{11}$$

The boundary condition is

$$\frac{\lambda_\infty^+}{\lambda_\infty^-} = e^{2h} . \tag{12}$$

We can also obtain the recursion relations of f_s^{++}, f_s^{+-} and f_s^{--}. From the structure of NT_D we note that the path between the branching points s and $s+1$ can be regarded as composed by two identical parts, each of them equal to the path between the branching points $s-1$ and s. Thus we obtain

$$f_s^{++} = e^h(f_{s-1}^{++})^2 + e^{-h}(f_{s-1}^{+-})^2 \tag{13}$$

$$f_s^{+-} = e^h(f_{s-1}^{++})(f_{s-1}^{+-}) + e^{-h}(f_{s-1}^{+-})(f_{s-1}^{--}) \tag{14}$$

$$f_s^{--} = e^h(f_{s-1}^{+-})^2 + e^{-h}(f_{s-1}^{--})^2 \tag{15}$$

and

$$\frac{f_s^{++}}{f_s^{+-}} = \frac{e^{2h}\left(\frac{f_{s-1}^{++}}{f_{s-1}^{+-}}\right)^2 + 1}{e^{2h}\frac{f_{s-1}^{++}}{f_{s-1}^{+-}} + \frac{f_{s-1}^{--}}{f_{s-1}^{+-}}} \tag{16}$$

$$\frac{f_s^{--}}{f_s^{+-}} = \frac{\left(\frac{f_{s-1}^{--}}{f_{s-1}^{+-}}\right)^2 + e^{2h}}{e^{2h}\frac{f_{s-1}^{++}}{f_{s-1}^{+-}} + \frac{f_{s-1}^{--}}{f_{s-1}^{+-}}} . \tag{17}$$

The initial condition (corresponding to the path between points 0 and 1) is

$$f_0^{++} = e^K \qquad f_0^{+-} = e^{-K} \qquad f_0^{--} = e^k . \tag{18}$$

We note that in the recursion relation (11) f_s^{++}/f_s^{+-} and f_s^{--}/f_s^{+-} are the functions of s which prevent us from applying the analytic method used for Cayley trees [12]. We have to make a numerical calculation.

Combining the essential equations (5), (8), (11), (16) and (17), the boundary condition (12) and the initial condition (18), we finally get $m_0 = 0$ (see the appendix). By the same procedure, we can easily obtain that the spontaneous magnetizations of all other sites are also zero and so is the mean value.

3. Conclusion and discussion

So far we have got that the free energy of the system is analytic for all finite T when there is no applied field and the spontaneous magnetizations of all sites are zero. We conclude that the phase transition on NT_D is a trivial one and different from that on Cayley trees. The reason for this difference may be simply understood as in contrast to the situation on Cayley trees where the distances between neighbouring branching points are all equal, the distances on NT_D grows very fast (which is 2^s between the branching points s and $s+1$) as s increases, and thus the phase transition on NT_D is, in some sense, very like that on a one-dimensional chain.

Our result is independent of the d_S of the structure and gives an example of a trivial phase transition at $d_s \geqslant 2$ which does not agree with the argument of Yu and Gong [8]. We note that the periodic Koch lattice (PKL) which was used as the example of a finitely ramified fractal with $d_s = 2$ in [8] is, in fact, an infinitely ramified one, for PKL can be straightened to the square lattice and hence has the same infinite ramification as the latter.

We believe that d_s plays an important role in determining phase transition. However, our results suggest that the role may be complex and further work is needed.

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Appendix. The calculation of m_0

We define $M(L)$ the spontaneous magnetization of the point o of a finite NT_D, where L is the largest label of the branching points. In the limit of $L \to \infty$, $M(L) = m_0$.

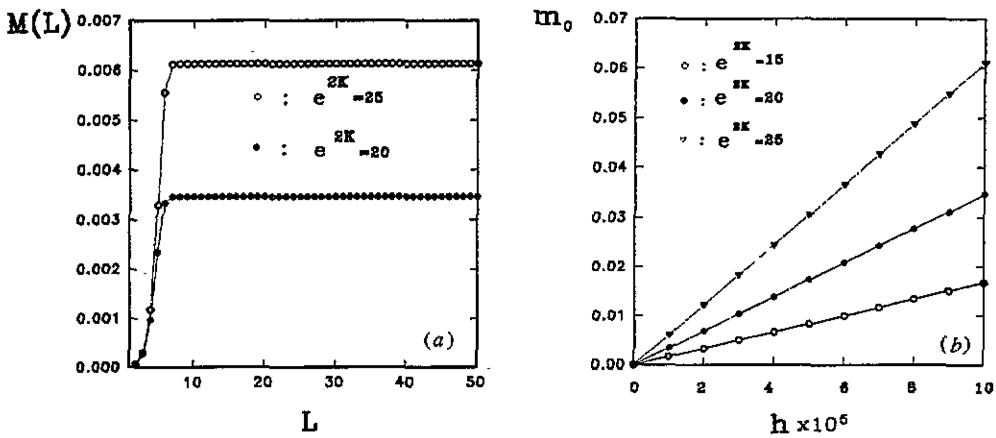

Figure A1. *(a) The spontaneous magnetization of the point o $M(L)$ versus L. $M(L) = m_0$ in the limit of $L \to \infty$. It shows that $M(L)$ quickly approaches to its asymptotic value as L increases. The parameters are $h = 0.000\,01$ and $e^{2K} = 20, 25$. (b) The asymptotic value m_0 (approximated by $M(2000)$ versus h). It shows that in the limit of $h \to 0$, m_0 equals zero. The parameters are $e^{2K} = 15, 20$ and 25.*

The boundary condition (12) is changed as

$$\frac{\lambda_L^+}{\lambda_L^-} = e^{2h} . \tag{A1}$$

Combining the essential equations (5), (8), (11), (16) and (17), the boundary condition (19) and the initial condition (18), we obtain $M(L)$, as shown in figure A1.

In figure A1(a) we show that $M(L)$ approaches its asymptotic value m_0 very quickly as L increases.

In figure A1(b) we show that the asymptotic value m_0 equals zero when $h \to 0$, which is independent of the coupling parameter K.

So we conclude $m_0 = 0$ on NT_D.

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